On the structure of inner set mappings

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Let S be a given set of power $\mathfrak{m}$, I_1 and I_2 two arbitrary classes of subsets of S. A function $G(X)$ is called a set mapping if $G(X)$ is defined on I_1 and such that, for each $X \in I_1$, $G(X) \in I_2$. We say that $G(X)$ is an *inner set mapping* if, for each $X \in I_1$, $G(X) \subset X$. Let further $X_0 \in I_2$, we define the inverse of X_0 in two different ways, first as the set

$$\bigcup_{G(X)=X_0} X = \overline{\overline{X_0^{-1}}}$$

and second as the set

$$\{X : G(X) = X_0\} = X_0^{*-1}.$$

The set of all subsets of power $\mathfrak{n}$ and the set of all subsets of power $<\mathfrak{n}$ of S are denoted by $[S]^{\mathfrak{n}}$ and $[S]^{<\mathfrak{n}}$, respectively. If $I_1 = [S]^{\mathfrak{n}}$ or $I_1 = [S]^{<\mathfrak{n}}$, then a set mapping defined on $I_1 = [S]^{\mathfrak{n}}$ or $I_1 = [S]^{<\mathfrak{n}}$ is called a set mapping of type $\mathfrak{n}$ or type $<\mathfrak{n}$, respectively. If for a set mapping $G(X)$ is $I_2 = [S]^{\mathfrak{n}}$ or $I_2 = [S]^{<\mathfrak{n}}$, then $G(X)$ is called a set mapping of range $\mathfrak{n}$ or range $<\mathfrak{n}$, respectively.

We introduce now the symbols $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{r}$. These symbols indicate that for every set mapping of the type $\mathfrak{q}$ and range $\mathfrak{p}$, defined on the set S of power $\mathfrak{m}$, there exists an element $X_0 \in [S]^{\mathfrak{p}}$ for which $\overline{\overline{X_0^{-1}}} = \mathfrak{r}$ or $\overline{\overline{X_0^{*-1}}} = \mathfrak{r}$, respectively. The symbol $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \to \mathfrak{r}$ has an analogous meaning. The same symbols, with $\to$ replaced by $\not\to$, indicate the negation of the corresponding statement.

It is obvious, that we have to suppose $\mathfrak{m} \geqq \mathfrak{q} \geqq \mathfrak{p}$. We prove in this paper the following results:

a) *negative results* ($\mathfrak{q} \geqq \aleph_0$):

1) if $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}}$, then $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \not\to \mathfrak{q}^+$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \not\to 2$,

2) if $\mathfrak{p} = \mathfrak{q}$, then $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \not\to \mathfrak{q}^+$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \not\to 2$.

b) *positive results* ($\mathfrak{q} \geqq \aleph_0$):

1) $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$ if $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$,

2) $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ if $\mathfrak{q}^{\mathfrak{p}} < (\mathfrak{m}^{\mathfrak{q}})^*$ and $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{q}}$.

These results make possible with the aid of the generalized continuum hypothesis, the discussion in almost every case. We can obviously assume, that $\mathfrak{p} < \mathfrak{q}$ and $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}}$. Thus we can state:

c) $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{m}$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^{*} \rightarrow \mathfrak{m}^{\mathfrak{q}}$, if $\mathfrak{q}^{\mathfrak{p}} \neq \mathfrak{m}^{*}$ or $\mathfrak{q} \geqq \mathfrak{m}^{*}$. Thus the only open question is the following:

Is it true, that $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{m}$ or $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^{*} \rightarrow \mathfrak{m}^{\mathfrak{q}}$ if $\mathfrak{m} = \aleph_{\alpha}$, α is of second kind, $\mathfrak{q} = \aleph_{cf(\alpha)-1}$, $cf(\alpha) - 1$ is of second kind and $\mathfrak{p} = \aleph_{\beta}$ with $\beta \geqq cf(cf(\alpha) - 1)$?; for instance in the simplest case:

$$((\aleph_{\omega_{\omega+1}}, \aleph_0, \aleph_{\omega})) \rightarrow \aleph_{\omega_{\omega+1}} ?$$

or

$$((\aleph_{\omega_{\omega+1}}, \aleph_0, \aleph_{\omega}))^{*} \rightarrow \aleph_{\omega_{\omega+1}}^{\aleph_{\omega}} = \aleph_{\omega_{\omega+1}} ?$$

d) if $0 < k < l < \infty$, then $((\aleph_{\alpha+k}, k, l)) \rightarrow \aleph_{\alpha}$;
if $0 < k < l < \infty$, then $((\aleph_{\alpha+k}, k, l)) \not\rightarrow \aleph_{\alpha+1}$.

e) Finally we deal with the symbol $((\mathfrak{m}, < \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$. If $\mathfrak{p} < \mathfrak{q}$, then the validity of the symbol $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$ implies the validity of $((\mathfrak{m}, < \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$. This holds in the case too, if $\mathfrak{p} = \mathfrak{q}$ and $\mathfrak{q} = \aleph_{\alpha}$ has an index of first kind. If $\mathfrak{q}$ is regular, $\mathfrak{q} \geqq \aleph_0$, and $\mathfrak{r}^{\mathfrak{n}} < \mathfrak{m}^{*}$ for every $\mathfrak{r} < \mathfrak{q}$ and $\mathfrak{n} < \mathfrak{q}$, then $((\mathfrak{m}, < \mathfrak{q}, \mathfrak{q})) \rightarrow \mathfrak{m}$; thus in particular $((\mathfrak{m}, < \aleph_0, \aleph_0)) \rightarrow \mathfrak{m}$. The simplest unsolved problem with respect to the symbol $((\mathfrak{m}, < \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$ is the following:

$$((\aleph_{\omega+2}, < \aleph_{\omega}, \aleph_{\omega})) \rightarrow \aleph_{\omega+1} \text{ or } \aleph_{\omega+2} ?$$

Set mappings of type 1 and range $\mathfrak{n}$ or $< \mathfrak{n}$ have been investigated previously in [1], [2], [3], [4].

Notations and definitions. Throughout this paper, the symbols $\overline{\overline{S}}$ and $\bar{\beta}$ denote the cardinal number of S and the ordinal number β, respectively. For any cardinal number $\mathfrak{r} (= \aleph_{\alpha})$ we denote by $\varphi_{\mathfrak{r}}$ the initial number of $\mathfrak{r}$, by $\mathfrak{r}^{*}$ the smallest cardinal number for which $\mathfrak{r}$ is the sum of $\mathfrak{r}^{*}$ cardinal numbers each of which is smaller than $\mathfrak{r}$, by $cf(\alpha)$ the index β of the initial number ω_{β} of $\mathfrak{r}^{*}$, by $\mathfrak{r}^{+}$ the cardinal number immediately following $\mathfrak{r}$.

I.

We prove now negative results with respect to the symbols $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$ and $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^{*} \rightarrow \mathfrak{r}$. First we prove the following:

Theorem 1. *Let* $\mathfrak{p}$, $\mathfrak{q}$ *and* $\mathfrak{m}$ *be cardinal numbers such that* $\mathfrak{m} \geqq \mathfrak{q} \geqq \mathfrak{p} \geqq \aleph_0$. *If* $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}} = \mathfrak{r}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \not\rightarrow \mathfrak{q}^{+}$.

Proof. Let $\overline{\overline{S}} = \mathfrak{m}$. We define on S a one to one set mapping $G(X)$ of type $\mathfrak{q}$ and range $\mathfrak{p}$ which shows that the theorem is true. By the hypothesis

$$\overline{\overline{[S]^{\mathfrak{q}}}} = \mathfrak{r}.$$

Let

$$X_0, X_1, \ldots, X_\omega, X_{\omega+1}, \ldots, X_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{r}})$$

be a well-ordering of the set $[S]^{\mathfrak{q}}$ of the type $\varphi_{\mathfrak{r}}$. We define $G(X)$ by transfinite induction as follows. Let $G(X_0)$ be an arbitrary subset of X_0 of power $\mathfrak{p}$, and ν a given ordinal number, $0 < \nu < \varphi_{\mathfrak{r}}$. Suppose that all sets $G(X_\mu)$, where $0 \leqq \mu < \nu$, have been already defined such that

1) $\overline{\overline{G(X_\mu)}} = \mathfrak{p}$, for $\mu < \nu$,
2) $G(X_\mu) \subset X_\mu$, for $\mu < \nu$,
3) $G(X_{\mu_1}) \neq G(X_{\mu_2})$ for $\mu_1 < \mu_2 < \nu$.

Since the power of the set $[X_\nu]^{\mathfrak{p}}$ is $\mathfrak{r}$ too, there exists a subset of X_ν of power $\mathfrak{p}$ which is distinct from each $G(X_\mu)$ with index $\mu < \nu$, because $\nu < \varphi_{\mathfrak{r}}$. Let $G(X_\nu)$ be such a subset of X_ν. Then $\overline{\overline{G(X_\nu)}} = \mathfrak{p}$, $G(X_\nu) \subset X_\nu$ and $G(X_\mu) \neq G(X_\nu)$ for $\mu < \nu$. Thus $G(X)$ is defined for every element of $[S]^{\mathfrak{q}}$ and it is a one to one inner set mapping of type $\mathfrak{q}$ and range $\mathfrak{p}$. The theorem is proved.

Corollary 1. *If* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every* β, *then* $((\aleph_{\omega_\alpha+1}, \aleph_\alpha, \aleph_{\omega_\alpha})) \not\rightarrow \aleph_{\omega_\alpha+1}$.

It follows from the proof of Theorem 1 also the following

Theorem 2. *Let* $\mathfrak{p}$, $\mathfrak{q}$ *and* $\mathfrak{m}$ *be cardinal numbers such that* $\mathfrak{m} \geqq \mathfrak{q} \geqq \mathfrak{p} \geqq \aleph_0$. *If* $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{q}^{\mathfrak{p}}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^{*} \not\rightarrow 2$.

Theorem 3. *If* $\mathfrak{q} \geqq \aleph_0$, *then* $((\mathfrak{m}, \mathfrak{q}, \mathfrak{q})) \not\rightarrow \mathfrak{q}^{+}$ *for every cardinal number* $\mathfrak{m} > \mathfrak{q}$.

Instead of Theorem 3 we prove the following stronger result:

Theorem 4. *Let S be a set of power* $\mathfrak{m} > \mathfrak{q}$. *There exists a function* $G(X)$ *defined on* $[S]^{\mathfrak{q}}$ *with the following properties:*

(1) $G(X) \subset X$ *and* $X - G(X) \neq 0$ *for every* $X \in [S]^{\mathfrak{q}}$

(2) $G(X) \in [S]^{\mathfrak{q}}$ *for every* $X \in [S]^{\mathfrak{q}}$;

(3) $G(X) \neq G(Y)$ *if X and Y are two distinct elements of* $[S]^{\mathfrak{q}}$;

(4) *for every* $Y \in [S]^{\mathfrak{q}}$ *there exists an element* $X \in [S]^{\mathfrak{q}}$ *such that* $Y = G(X)$.[1])

Proof. Let E be a set of power $\mathfrak{n} \geqq \mathfrak{q}$; we prove that there exists a a function $F(X)$ defined on $[E]^{\mathfrak{q}}$ which satisfies the conditions (1), (2), and (3).

We consider two cases: (i) $\overline{\overline{E}} = \mathfrak{q}$, and (ii) $\overline{\overline{E}} > \mathfrak{q}$.

Ad (i). Let

$$X_0, X_1, \ldots, X_\omega, \ldots, X_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{r}})$$

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[1]) For the proof of Theorem 1 it is sufficient that $G(X)$ satisfy the conditions (1), (2), and (3). This theorem is proved in [5].

be a well-ordering of $[E]^{\mathfrak{q}}$ of the type $\varphi_{\mathfrak{r}}$, where $\mathfrak{r}=2^{\mathfrak{q}}$. We define $F(X)$ by transfinite induction as follows. Let $F(X_0)$ be an arbitrary proper subset of X_0 of power $\mathfrak{q}$, and β a given ordinal number, $0<\beta<\varphi_{\mathfrak{r}}$. Suppose that all sets $F(X_\xi)$, where $0 \leqq \xi < \beta$, have been already defined such that the conditions (1), (2), (3) are satisfied. Since the power of the set $[X_\beta]^{\mathfrak{q}}$ is $2^{\mathfrak{q}}$, and $\overline{\beta}<2^{\mathfrak{q}}$, there is a subset Y of X_β, of power $\mathfrak{q}$, such that $X_\beta - Y \neq 0$ and Y is distinct from each $F(X_\xi)$ with index $\xi<\beta$. Let $F(X_\beta)=Y$. Thus $F(X)$ is defined for every element of $[E]^{\mathfrak{q}}$ such that the conditions (1), (2), and (3) are satisfied.

Ad (ii) Consider the set $\mathbf{M}$ of all subsets M of $[E]^{\mathfrak{q}}$ such that if X and Y are two distinct elements of M then $\overline{\overline{X \cap Y}}<\mathfrak{q}$. By ZORN's Lemma there is a maximal element M_0 of $\mathbf{M}$. Let

$$Z_0, Z_1, \ldots, Z_\omega, Z_{\omega+1}, \ldots, Z_\xi, \ldots \qquad (\xi<\varphi_{\mathfrak{i}})$$

be a well-ordering of M_0 of the type $\varphi_{\mathfrak{i}}$, where $\mathfrak{i}=\overline{\overline{M}}_0$. Since $\overline{\overline{Z}}_\xi=\mathfrak{q}$ for every $\xi<\varphi_{\mathfrak{i}}$, there exists a function $F_\xi(Z)$ on $[Z_\xi]^{\mathfrak{q}}$ which satisfies the conditions (1), (2), and (3). Let now $X \in [E]^{\mathfrak{q}}$. By the definition of M_0 there is a smallest ordinal number $\nu=\nu(X)$ for which $\overline{\overline{X \cap Z_\nu}}=\mathfrak{q}$. Let

$$F(X)=F_{\nu(X)}(X \cap Z_{\nu(X)}) \cup (X - Z_{\nu(X)}).$$

It is obvious that $F(X)$ satisfies the conditions (1) and (2). For the proof of (3) let $Y \neq X$ be another element of $[E]^{\mathfrak{q}}$ Then

$$F(Y)=F_{\nu(Y)}(Y \cap Z_{\nu(Y)}) \cup (Y - Z_{\nu(Y)}).$$

There are two cases: 1) $\nu(X)=\nu(Y)$, 2) $\nu(X)\neq\nu(Y)$.

Ad 1. If $X \cap Z_{\nu(X)} \neq Y \cap Z_{\nu(X)}$, then by the definition of $F_{\nu(X)}$ $F_{\nu(X)}(X \cap Z_{\nu(X)}) \neq F_{\nu(X)}(Y \cap Z_{\nu(X)})$. We may assume that $F_{\nu(X)}(Y \cap Z_{\nu(X)})$ does not contain $F_{\nu(X)}(X \cap Z_{\nu(X)})$. Let $x_0 \in F_{\nu(X)}(X \cap Z_{\nu(X)})$ such that $x_0 \notin F_{\nu(X)}(Y \cap Z_{\nu(X)})$. By the condition $F_{\nu(X)}(Z) \subset Z$, we have that $x_0 \in X \cap Z_{\nu(X)}$. It follows that $x_0 \notin Y - Z_{\nu(X)}$; consequently $x_0 \notin F_{\nu(X)}(Y \cap Z_{\nu(X)}) \cup (Y - Z_{\nu(X)})$ i.e. $F(X) \neq F(Y)$.

If $X \cap Z_{\nu(X)} = Y \cap Z_{\nu(X)}$, then, since $X \neq Y$, $X - Z_{\nu(X)} \neq Y - Z_{\nu(X)}$; consequently, by the definition of F, $F(X) \neq F(Y)$.

Ad 2. We may suppose that $\nu(X)<\nu(Y)$. By the definition of M_0, $\overline{\overline{Z_{\nu(X)} \cap Z_{\nu(Y)}}}<\mathfrak{q}$, i.e. $\overline{\overline{(X \cap Z_{\nu(X)}) \cap (Y \cap Z_{\nu(Y)})}}<\mathfrak{q}$ consequently $F(X) \neq F(Y)$. Thus $F(X)$ statisfies the condition (3) too.

Let now F be a set of power $\mathfrak{r}>\mathfrak{q}$. It is easy see that there exists a function $H(X)$ on $[F]^{\mathfrak{q}}$ such that

a) $X \subset H(X)$ and $H(X)-X \neq 0$,

b) $\overline{\overline{H(X)}}=\mathfrak{q}$,

c) $H(X) \neq H(Y)$ if $X \neq Y$.

We apply now the following theorem of BANACH [6]: If the function φ maps the set A one to one onto a subset of B and the function ψ maps the set B one to one onto a subset of A, then there exists a decomposition $A = A_1 \cup A_2$ of A and a decomposition $B = B_1 \cup B_2$ of B such that $A_1 \cap A_2 = B_1 \cap B_2 = 0$, $\varphi(A_1) = B_1$ and $\psi(B_2) = A_2$.

Let now $A = B = [S]^{\mathfrak{q}}$ ($\overline{\overline{S}} = \mathfrak{m} > \mathfrak{q}$). Let further φ be a function on $[S]^{\mathfrak{q}}$ such that the conditions (1), (2), (3), and ψ a function on $[S]^{\mathfrak{q}}$ such that the conditions a), b), c) hold respectively. Then there exist two decompositions $[S]^{\mathfrak{q}} = A^1 \cup A_2 = B_1 \cup B_2$ such that $A_1 \cap A_2 = B_1 \cup B_2 = 0$, $\varphi(A_1) = B_1$ and $\psi(B_2) = A_2$. We define $G(X)$ on $[S]^{\mathfrak{q}}$ as follows. Let

$$G(X) = \begin{cases} \varphi(X), & \text{if } X \in A_1, \\ \psi^{-1}(X), & \text{if } X \in A_2. \end{cases}$$

Obviously $G(X)$ satisfies the conditions (1), (2), (3) and (4).

The proof of Theorem 4 gives also the following

Theorem 5. *If* $\mathfrak{q} \geqq \aleph_0$, *then* $((\mathfrak{m}, \mathfrak{q}, \mathfrak{q}))^* \not\rightarrow 2$.

II.

We assume in this chapter that $\mathfrak{p} < \mathfrak{q}$, $\mathfrak{q} \geqq \aleph_0$ and $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}}$ and prove:

Theorem 6. *If* $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{m}$.

Proof. Suppose that the theorem is not true, i.e. for every subset P of power $\mathfrak{p}$

$$\overline{\overline{\bigcup_{G(Q)=P} Q}} < \mathfrak{m}.$$

By the condition,

$$\overline{\overline{\bigcup_{P' \subseteq P} \bigcup_{G(Q)=P'} Q}} < \mathfrak{m}$$

for every subset P of S of power $\mathfrak{p}$.

We define now by transfinite induction a sequence $\{P_\xi\}_{\xi < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{q}+}}$ of the type $\varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$ of the subsets of S of power $\mathfrak{p}$ as follows. Let P_0 be an arbitrary subset of S of power $\mathfrak{p}$ and β a given ordinal number, $0 < \beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$. Suppose that all sets P_ξ, where $0 \leqq \xi < \beta$, have been already defined, and let $A_\beta = \bigcup_{\xi < \beta} P_\xi$. Since $\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}+}$ and $\overline{\overline{P_\xi}} = \mathfrak{p} < \mathfrak{q}$ we have $\overline{\overline{A_\beta}} \leqq \mathfrak{q}$. It follows by the hypothesis $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$ that

$$\overline{\overline{\bigcup_{P \subseteq A_\beta} \bigcup_{G(Q)=P} Q}} < \mathfrak{m}.$$

We define the set P_β as a subset of power $\mathfrak{p}$, of the set

$$S - \bigcup_{\xi < \beta} P_\xi - \bigcup_{P \subseteq A_\beta} \bigcup_{G(Q)=P} Q.$$

Put

$$(1) \qquad H = \bigcup_{\xi < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}} P_\xi .$$

It is obvious that $\overline{\overline{H}} = \mathfrak{q}$. It follows that there exists a subset P of H of power $\mathfrak{p}$ such that $G(H) = P$. Since $\mathfrak{p}^+$ is regular there exists an ordinal number $\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}$ such that

$$P \subseteq \bigcup_{\xi < \beta} P_\xi = A_\beta .$$

But then clearly by the definition of P_β, $P_\beta \not\subseteq H$, which contradicts (1).

Corollary 2. *If* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every* β, *then* $((\aleph_{\omega_\alpha+2}, \aleph_\alpha, \aleph_{\omega_\alpha})) \to \to \aleph_{\omega_\alpha+2}$.

Theorem 7. *If* $\mathfrak{p} < \mathfrak{q}^*$ *and* $\mathfrak{r}^{\mathfrak{p}} < \mathfrak{m}^*$ *for every* $\mathfrak{r} < \mathfrak{q}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$.

The proof of Theorem 7 is similar to the proof of Theorem 6.

Remark. If $\mathfrak{q} < \mathfrak{m}^*$, then $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$, because if $\mathfrak{q} = \aleph_\alpha$ with index α of second kind or $\aleph_{\alpha+1} = \mathfrak{q}$, then

$$\sum_{\mathfrak{r} < \mathfrak{q}} \mathfrak{r}^{\mathfrak{p}} = \mathfrak{q}^{\mathfrak{p}} \quad \text{or} \quad \sum_{\mathfrak{r} < \mathfrak{q}} \mathfrak{r}^{\mathfrak{p}} = \aleph_\alpha^{\mathfrak{p}},$$

respectively, i. e. in this case Theorem 7 is a particular case of Theorem 6.

Corollary 2. *If* $\mathfrak{q} = \mathfrak{m}^*$ *and* $\mathfrak{r}^{\mathfrak{p}} < \mathfrak{m}^*$ *for every* $\mathfrak{r} < \mathfrak{q}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$.

Corollary 3. *If* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every* β, $\mathfrak{m}^* = \mathfrak{q} = \aleph_{\alpha+1}$ *and* $\mathfrak{p} < (\aleph_\alpha)^*$, *then* $(\mathfrak{m}, \mathfrak{p}, \mathfrak{q}) \to \mathfrak{m}$.

Theorem 8. *Let* $\mathfrak{p}$, $\mathfrak{q}$ *and* $\mathfrak{m}$ *be cardinal numbers such that* $\mathfrak{m} \geqq \mathfrak{q}$. *If* $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{m}^{\mathfrak{p}}$ *and* $\mathfrak{q}^{\mathfrak{p}} < (\mathfrak{m}^{\mathfrak{q}})^*$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$.

Proof. The proof of this theorem is similar to the proof of Theorem 6. Suppose that the theorem is not true, i. e. for every subset P of S of power $\mathfrak{p}$, the power of the set

$$P^{*-1} = \{Q : G(Q) = P\}$$

is smaller than $\mathfrak{m}$. Let $\Gamma(P)$ be the set of all sets $P' \in [S]^{\mathfrak{p}}$ for which there exists a set $Q \in [S]^{\mathfrak{q}}$ such that $G(Q) = P_0$ for some $P_0 \subseteq P$ and $P' \subset Q$. Then by the condition

$$\overline{\overline{\Gamma(P)}} < \mathfrak{m}^{\mathfrak{q}}$$

for every subset P of S of power $\mathfrak{p}$.

We define now by transfinite induction a sequence $\{P_\xi\}_{\xi < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}}$ of the type $\varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}$ of the sets $\in [S]^{\mathfrak{p}}$ as follows. Let P_0 be an arbitrary element of $[S]^{\mathfrak{p}}$ and β a given ordinal number, $0 < \beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}$. Suppose that all sets P_ξ, where $0 \leqq \xi < \beta$, have been already defined, and let $A_\beta = \bigcup_{\xi < \beta} P_\xi$. Since

$\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}$ and $\overline{\overline{P_\xi}} = \mathfrak{p} < \mathfrak{q}$, we have $\overline{\overline{A_\beta}} \leqq \mathfrak{q}$. It follows by the hypothesis $\mathfrak{q}^{\mathfrak{p}} < (\mathfrak{m}^{\mathfrak{q}})^*$ that

$$\overline{\overline{\bigcup_{P \subseteqq A_\beta} \Gamma(P)}} < \mathfrak{m}^{\mathfrak{q}}.$$

We define the set P_β as a subset of power $\mathfrak{p}$, of the set

$$[S]^{\mathfrak{p}} - \{P_\xi\}_{\xi<\beta} - \bigcup_{P \subseteqq A_\beta} \Gamma(P).$$

Since $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{q}}$, there exists such an element of $[S]^{\mathfrak{p}}$. Put

$$H = \bigcup_{\xi < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}} P_\xi. \tag{2}$$

It is obvious that $\overline{\overline{H}} = \mathfrak{q}$. It follows that there exists a subset P of H of power $\mathfrak{p}$ such that $G(H) = P$. Since $\mathfrak{p}^+$ is regular there exists an ordinal number $\beta < \varphi_{\mathfrak{q}} + \varphi_{\mathfrak{p}^+}$ such that

$$P \subseteq \bigcup_{\xi<\beta} P_\xi = A_\xi.$$

But then clearly, by the definition of P_β, $P_\beta \subseteq\mathrel{\mkern-3mu}\not\equiv H$, which contradicts (2).

III.

We assume in this chapter that $\mathfrak{p} < \mathfrak{q}$, $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}}$ and the generalized continuum hypothesis holds, i. e. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal number α.

Lemma. *If* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}$. We omit the proof.

Theorem 9. *If* $\mathfrak{q}^{\mathfrak{p}} \neq \mathfrak{m}^*$ *or* $\mathfrak{q} \geqq \mathfrak{m}^*$, *then* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ *and* $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$.

Proof. Suppose first, that $\mathfrak{q}^{\mathfrak{p}} \neq \mathfrak{m}^*$. Thus if $\mathfrak{q}^{\mathfrak{p}} < \mathfrak{m}^*$, then $((\mathfrak{m}, \mathfrak{p}, \mathfrak{q})) \to \mathfrak{m}$ by Theorem 6 and $(\mathfrak{m}, \mathfrak{p}, \mathfrak{q}))^* \to \mathfrak{m}^{\mathfrak{q}}$ by the Lemma and Theorem 6, because in this case $\mathfrak{m}^{\mathfrak{q}} = \mathfrak{m}$.

If $\mathfrak{q}^{\mathfrak{p}} > \mathfrak{m}^*$, then we consider two cases: a) $\mathfrak{p} < \mathfrak{m}^*$ and b) $\mathfrak{p} \geqq \mathfrak{m}^*$.

Ad a. We have in this case that $\mathfrak{q} \geqq \mathfrak{m}^*$. It follows that $\mathfrak{m} = \mathfrak{m}^{\mathfrak{p}} < \mathfrak{m}^{\mathfrak{q}} = \mathfrak{m}^+$; therefore there exists a set P_0 in $[S]^{\mathfrak{p}}$ for which $\overline{\overline{P_0^{*-1}}} = \mathfrak{m}^{\mathfrak{p}}$ and consequently $\overline{\overline{P_0^{*-1}}} = \mathfrak{m}$.

Ad b. We have in this case that $\mathfrak{q} \geqq \mathfrak{m}^*$; consequently $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}$. It follows that $\mathfrak{m}^{\mathfrak{q}} = (\mathfrak{m}^{\mathfrak{q}})^*$ Since the assumptions of Theorem 8 hold, there exists a set P_0 in $[S]^{\mathfrak{p}}$ such that $\overline{\overline{P_0^{*-1}}} = \mathfrak{m}^+$ i. e. $\overline{\overline{P_0^{-1}}} = \mathfrak{m}$.

Finally if $\mathfrak{q}^{\mathfrak{p}} = \mathfrak{m}^*$, then $\mathfrak{q} \geqq \mathfrak{m}^*$ by the assumption, and if in this case $\mathfrak{p} < \mathfrak{m}^*$, then the proof is the same as in the case a) while if $\mathfrak{p} \geqq \mathfrak{m}^*$, then our statement follows from Theorem 8.

IV.

We assume now that $\mathfrak{p}$ and $\mathfrak{q}$ are finite cardinal numbers and we prove

Theorem 10. *If k and l are two natural numbers such that $0 < k < l$, then $((\aleph_{\alpha+k}, k, l)) \to \aleph_\alpha$ for every ordinal number α.*

Proof. We use induction with respect to k. Let $k = 1$ and $l > 1$. Suppose that the theorem is false, i. e., for every element

$$\overline{\overline{\bigcup_{G(P)=\{x\}} P}} < \aleph_\alpha. \tag{3}$$

Let F be a subset of S of the power $\aleph_\alpha$ and omit from the set the elements of the set

$$H = \bigcup_{x \in F} \bigcup_{G(P)=\{x\}} P.$$

Since $\overline{\overline{F}} = \aleph_\alpha$, it follows from (3) that $\overline{\overline{S-H}} = \aleph_{\alpha+1}$. Let x_0 be an arbitrary element of $S-H$. If $\{x_0, y_1, \ldots, y_{l-1}\}$ is a set of l elements such that $\{y_1, y_2, \ldots, y_{l-1}\} \subset F$, then $G(\{x_0, y_1, \ldots, y_{l-1})\} = \{x_0\}$, for if not then $G(\{x_0, y_1, \ldots, y_{l-1}\}) = \{y_n\}$ for some n, $1 \leqq n \leqq l-1$. In this case $x_0 \in H$, which is a contradiction. Thus, since $\overline{\overline{H}} = \aleph_\alpha$,

$$\overline{\overline{\bigcup_{G(P)=\{x_0\}} P}} = \aleph_\alpha,$$

which contradicts (3). The theorem is proved in the case $k = 1$.

Suppose now that $k > 1$ and the theorem is true for $k-1$. Let F be a subset of S, of power $\aleph_{\alpha+k-1}$. Let $\mathfrak{L}$ be the set of all subsets L of S, of l elements, such that

$$\overline{\overline{L \cap (S-F)}} = 1.$$

We have two cases:

1) $\mathfrak{L}$ has a subset $\mathfrak{L}'$ of power $\aleph_{\alpha+k}$ such that $G(L) \subset F$ for every $L \in \mathfrak{L}'$.

2) For every subset L of $[F]^{l-1}$ the power of the set of the elements $x \in S-F$ for which $G(L \cup \{x\}) \subset F$, is smaller than $\aleph_{\alpha+k}$.

Ad 1. Since the power of the set $[F]^{l-1}$ is $\aleph_{\alpha+k-1}$ there exists an element L_0 of $[F]^{l-1}$ and a subset B of $S-F$ of power $\aleph_{\alpha+k}$ such that

$$G(L_0 \cup \{x\}) \subset L_0$$

for every $x \in B$. It follows that there exists a subset K_0 of k elements and a subset B' of B of power $\aleph_{\alpha+k}$ such that

$$G(L_0 \cup \{x\}) = K_0$$

for every $x_0 \in B'$. But then

$$\overline{\overline{\bigcup_{G(L)=K_0} L}} = \aleph_{\alpha+k}.$$

Ad 2. Since $\aleph_{\alpha+k}$ is regular $S-F$ has an element x_0 such that

$$x_0 \in G(L \cup \{x_0\})$$

for every element L of $[F]^{l-1}$. We define now an inner set mapping $F(X)$ on $[F]^{l-1}$ into $[F]^{k-1}$ as follows. Let

$$F(L) = G(L \cup \{x_0\}) - \{x_0\}$$

for every $L \in [F]^{l-1}$. It is obvious that $F(L) \subset L$. By the induction hypothesis for $k-1$ the theorem is true, i. e. there is an element K of $[F]^{k-1}$ such that

$$\overline{\overline{\bigcup_{F(L)=K} L}} = \aleph_\alpha.$$

It follows from the definition of $F(X)$ that

$$\overline{\overline{\bigcup_{G(L)=K\cup\{x_0\}} L}} = \aleph_\alpha.$$

which proves the theorem.

Next we show that Theorem 5 cannot be improved.

Theorem 11. *If k and l natural numbers, $0 < k < l$, then* $((\aleph_{\alpha+k}, k, l)) \dashv\!\!\!\rightarrow \aleph_{\alpha+1}$.

Proof. Let S be a set of power $\aleph_{\alpha+k}$ and

(4) $$x_0, x_1, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \qquad (\xi < \omega_{\alpha+k})$$

a well-ordering of S of type $\omega_{\alpha+k}$. We define now an inner set mapping $G(X)$ of type k and range l as follows. Let L be an arbitrary element of $[S]^l$, and x_{ξ_1} the greatest element of L in the series (4). Let further

(5) $$x_0^{\xi_1}, x_1^{\xi_1}, \ldots, x_\omega^{\xi_1}, \ldots, x_{\omega+1}^{\xi_1}, \ldots, x_\xi^{\xi_1}, \ldots, \qquad (\xi < \omega(\xi_1))$$

be a well-ordering of the set $\{x_\mu\}_{\mu<\xi_1}$, where $\omega(\xi_1)$ is the initial number of $\bar{\xi}_1$. Let now $x_{\xi_2}^{\xi_1}$ be the greatest element of $L-\{x_{\xi_1}\}$ in the series (5) and let $\{x_\xi^{\xi_1,\xi_2}\}_{\xi<\omega(\xi_2)}$ be a well-ordering of the subset $\{x_\xi^{\xi_1}\}_{\xi<\xi_2}$ of (5), where $\omega(\xi_2)$ is the initial number of $\bar{\xi}_2$. Suppose that the element $x_{\xi_n}^{\xi_1,\ldots,\xi_{n-1}}$ and the series $\{x_\xi^{\xi_1,\ldots,\xi_n}\}_{\xi<\omega(\xi_n)}$ are defined for every n, $1 < n \leqq m < k$. We define now the element $x_{\xi_{m+1}}^{\xi_1,\xi_2,\ldots,\xi_m}$ as the greatest element of $L - \{x_{\xi_1}, x_{\xi_2}^{\xi_1}, x_{\xi_3}^{\xi_1,\xi_2}, \ldots, x_{\xi_m}^{\xi_1,\ldots,\xi_{m-1}}\}$ in the series $\{x_\xi^{\xi_1,\ldots,\xi_m}\}_{\xi<\omega(\xi_m)}$, where $\omega(\xi_m)$ is the initial number of $\bar{\xi}_m$. We define $G(L)$ as the set $\{x_{\xi_1}, x_{\xi_2}^{\xi_1}, \ldots, x_{\xi_k}^{\xi_1,\ldots,\xi_{k-1}}\}$. It is easy to see that for every element of $[S]^k$ the inverse has power $\leqq \aleph_\alpha$, which proves Theorem 9.

V.

We deal in this chapter with the symbol $((\mathfrak{m}, <\mathfrak{p}, \mathfrak{q})) \rightarrow \mathfrak{r}$.

Theorem 12. *Let $\mathfrak{q}$ and $\mathfrak{m}$ be two cardinal numbers such that $\mathfrak{q}$ is regular and $\mathfrak{q} \geqq \aleph_0$. If $\mathfrak{r}^{\mathfrak{n}} < \mathfrak{m}$ for every $\mathfrak{r} < \mathfrak{q}$ and $\mathfrak{n} < \mathfrak{q}$, then* $((\mathfrak{m}, <\mathfrak{q}, \mathfrak{q})) \rightarrow \mathfrak{m}$.

The proof of Theorem 12 is similar to the proof of Theorem 6.

Corollary 4. *If* $\mathfrak{q} = \aleph_0$ *or* $\mathfrak{q} > \aleph_0$ *is strongly inaccessible and* $\mathfrak{q} \leqq \mathfrak{m}^*$, *then* $((\mathfrak{m}, <\mathfrak{q}, \mathfrak{q})) \rightarrow \mathfrak{m}$.

Corollary 5. *Let* $2^{\aleph_\beta} = \aleph_{\beta+1}$ *for every* β. *If* $\aleph_\alpha$ *is regular and either* $\mathfrak{m} = \aleph_\alpha$ *or* $\aleph_\alpha < \mathfrak{m}^*$, *then* $((\mathfrak{m}, <\aleph_\alpha, \aleph_\alpha)) \rightarrow \mathfrak{m}$.

We can not prove that $((\mathfrak{m}, <\aleph_\omega, \aleph_\omega)) \rightarrow \mathfrak{n}$ *for some* $\mathfrak{m}$, *if* $\mathfrak{n} > \aleph_\omega$. If the generalized continuum hypothesis is true, then $((\aleph_{\omega+1}, <\aleph_\omega, \aleph_\omega)) \not\rightarrow \aleph_{\omega+1}$ (this is a consequence of Theorem 1).

Furthermore we are as yet not able to prove if $((\aleph_{\omega+2}, <\aleph_\omega, \aleph_\omega)) \rightarrow \aleph_{\omega+1}$ or if even $((\aleph_{\omega+2}, <\aleph_\omega, \aleph_\omega)) \rightarrow \aleph_{\omega+2}$?

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